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A Novel Approach for Solving Fuzzy Stochastic Data Envelopment Analysis Model in the Presence of Undesirable Outputs

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A B S T R A C T

Data Envelopment Analysis (DEA) is a widely used technique for measuring the relative efficiencies of Decision Making Units (DMUs) with multiple inputs and multiple outputs. However, Undesirable Outputs (UO) may be present in the production process which needs to be minimized. In real-world problems, the observed values of the input and output data are often vague or random. Indeed, Decision Makers (DMs) may encounter a hybrid uncertain environment where fuzziness and randomness coexist in a problem. This paper proposes fuzzy stochastic DEA model with undesirable outputs. The extensions to the fuzzy-stochastic environment sometimes may be laid to disregard some of the properties in DEA models such as linearity and feasibility. In this way, we apply a new version of DEA-UO model according to the probability-possibility approach to propose a linear and feasible model in deterministic form. A numerical example is presented to illustrate the features and the applicability of the proposed models.

1. Introduction

Data Envelopment Analysis (DEA), initially introduced by Charnes et al. [2], is a well-known non-parametric methodology for computing the relative efficiency of a set of homogeneous units, named as Decision Making Units (DMU). DEA generalizes the intuitive single-input single-output ratio efficiency measurement into a multiple-input multiple-output model by using a ratio of the weighted sum of outputs to the weighted sum of inputs. It computes scalar efficiency scores with a range of zero to one that determine efficient level or position for each DMU under evaluation among all DMUs. A DMU is said to be efficient if its efficiency score is equal to one, otherwise it is said to be inefficient. The basic DEA models were initially formulated only for desirable inputs and outputs. However, in real life problems, undesirable outputs may be present in the production process which also needs to be minimized. One of the direct approaches to deal with undesirable outputs is to treat all the desirable and undesirable outputs as the weighted sum, but using negative weights for the undesirable outputs.

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Hatami-Marbini et al. [12] classified the fuzzy DEA methods in the literature into five general groups, the tolerance approach (Sengupta [36], Triantis and Girod [41]), the α-level based approach, the fuzzy ranking approach (Guo and Tanaka [11], Hatami-Marbini et al. [13]), the possibility approach (Lertworasirikul et al. [21]), and the fuzzy arithmetic approach (Wang et al. [44]). Among these approaches, the α-level based approach is probably the most popular fuzzy DEA model in the literature. This approach generally tries to transform the FDEA model into a pair of parametric programs for each α-level. Kao and Liu [15], one of the most cited studies in the α-level approach's category, used Chen and Klein [3] method for ranking fuzzy numbers to convert the FDEA model to a pair of parametric mathematical programs for the given level of α. Saati et al. [34] proposed a fuzzy CCR model as a possibilistic programming problem and changed it into an interval programming problem by means of the α-level based approach. Puri and Yadav [32] applied the suggested methodology by Saati et al. [34] to solve fuzzy DEA model with undesirable outputs. Khanjani et al. [16] proposed fuzzy free disposal hull models under possibility and credibility measures. Momeni et al. [26] used fuzzy DEA models to address the impreciseness and ambiguity associated with the input and output data in supply chain performance evaluation problems. Payan [31] evaluated the performance of DMUs with fuzzy data by using the common set of weights based on a linear program.

In order to evaluate the efficiency of DMUs with the deterministic inputs and the random outputs, Land et al. [20] extended the chance constrained DEA model. Olesen and Petersen [28] developed the chance constrained programming (CCP) model for efficiency evaluation using a piecewise linear envelopment of confidence regions for observed stochastic multiple-input multiple-output combinations in DEA. Huang and Li [14] developed stochastic models in DEA by taking into account the possibility of random variations in input-output data. Cooper et al. [5], Li [21], and Bruni et al. [1] utilized joint chance constraints to extend the concept of stochastic efficiency. Cooper et al. [4] used chance-constrained programming for extending congestion DEA models. Tsionas and Papadakis [40] developed Bayesian inference techniques in chance-constrained DEA models. Udhayakumar et al. [43] used a genetic algorithm to solve the chance-constrained DEA models involving the concept of satisficing. Also some of the banking applications in relation to satisficing DEA can be found in Udhayakumar et al. [43] and Tsolas and Charles [42]. Farnoosh et al. [9] proposed chance-constrained FDH model with random input and random output. Wu et al. [45] proposed a stochastic DEA model by considering undesirable outputs with weak disposability. This model not only deals with the existence of random errors in the collected data, but also depicts the production rules uncovered by weak disposability of the undesirable outputs. A review of stochastic DEA models can be found in a recent work by Olesen and Petersen [29].

However, in the real-world problems decision makers may need to base decisions on information which are both fuzzily imprecise and probabilistically uncertain. Kwakernaak [18,19] introduced the concept of fuzzy random variable, and then this idea enhanced by a number of researchers in the literature (Feng and Liu [10], Liu and Liu [23], Liu [24], Qin and Liu [33]). Qin and Liu [33] developed a fuzzy random DEA (FRDEA) model where randomness and fuzziness exist simultaneously. The authors characterized the fuzzy random data with known possibility and probability distributions. Tavana et al. (2012) also introduced three different FDEA models consisting of probability-possibility, probability-necessity and probability-credibility constraints in which input and output data entailed fuzziness and randomness at the same time. Also, Tavana et al. [37] provided a chance-constrained DEA model with random fuzzy inputs and outputs with Poisson, uniform and normal distributions. After that, Tavana et al. [38] proposed DEA models with birandom input-output. Khanjani et al. [17] proposed fuzzy rough DEA models based on the expected value and possibility approaches. Paryab et al. [30] proposed DEA models using a bi-fuzzy data based possibility approach. However, there has been no attempt to study randomness and roughness simultaneously in DEA problems. To deal with the uncertain environments, especially hybrid environments, the DEA model may disorder its structure when the uncertain parameter of input and output exist. For example, the method proposed by Tavana et al. [39] does not compute the efficiency scores of DMUs in the range of zero to one for input-oriented DEA models. This study tries to overcome the shortcomings of the existing approach. Nasseri et al. [27] proposed a DEA model with undesirable output consisting of probability-possibility, probability-necessity and probability-credibility constraints. Ebrahimnejad et al. [7] formulated a deterministic linear model according to the probability–possibility approach for solving input-oriented fuzzy stochastic DEA model. Ebrahimnejad et al. [8] extended the concept of a normal distribution for fuzzy stochastic variables and propose a DEA model for problems characterized by fuzzy stochastic variables. To sum up with all the above aspects, the achievement of the present study is threefold: (1) to formulate a linear and feasible model with the efficiency scores of DUMs with the range of zero to one, (2) to propose a new version of the CCR-DEA model to achieve a linear and feasible model, and (3) to use a probability possibility approach for solving the uncertainty model.

The rest of this paper is organized as follows: In Section 2 the basic preliminaries with fuzzy sets and possibility set are given. Section 3 formulates the conventional DEA model in the presence of undesirable outputs. A probability- possibility approach is proposed for solving fuzzy stochastics DEA model in Section 5. A numerical example is given to illustrate the proposed approach in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

In this section, we review some necessary concepts related to the fuzzy set theory and probability theory, which will be used in the rest of paper [6, 46, 47].

Definition 1: A fuzzy set $\tilde{A}$, defined on universal set X, is given by a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$ where $\mu_{\tilde{A}}(x)$ gives the membership grade of the element x in the set $\tilde{A}$ and is called membership function.

Definition 2 : A fuzzy set $\tilde{A}$, defined on universal set of real numbers R, is said to be a fuzzy number if its membership function has the following characteristics:

1) $\tilde{A}$ is convex, i.e. $\forall x, y \in R, \forall \lambda \in [0,1], \mu_{\tilde{A}}(\lambda x + (1-\lambda) y) \geq min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$.

2) $\tilde{A}$ is normal, i.e. $\exists \bar{x} \in R; \mu_{\tilde{A}}(\bar{x}) = 1$.

3) $\mu_{\tilde{A}}$ is piecewise continuous.

Definition 3: A function $L:[0,\infty) \to [0,1]$ (or $R:[0,\infty) \to [0,1]$) is said to be reference function of fuzzy number if and only if $L(0)=1$ (or $R(0)=1$) and L or R is non-increasing on $[0,\infty)$.

Definition 4 [6]: A fuzzy number $\tilde{A} = (m, \alpha, \beta)_{LR}$ is said to be an LR fuzzy number, if its membership function is given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}), & \text{for} \quad x \leq m, \alpha > 0, \\ 1, & \text{for} \quad x = m, \\ R(\frac{x-n}{\beta}), & \text{for} \quad x \geqslant n, \beta > 0. \end{cases}$$

Remark 1: If $L(x) = R(x) = max\{0, 1-x\}$ then an LR fuzzy number $\tilde{A} = (m, \alpha, \beta)_{LR}$ is said to be a triangular fuzzy number and is denoted by $\tilde{A} = (m, \alpha, \beta)$.

Definition 5: Let $\tilde{A} = (m, \alpha, \beta)_{LR}$ be an LR fuzzy number and λ be a real number in the interval $[0,1]$ then the crisp set, $\tilde{A}_{\lambda} = \{x \in R : \mu_{\tilde{A}}(x) \geqslant \lambda\} = [A^L, A^R] = [m - \alpha L^{-1}(\lambda), m + \beta R^{-1}(\lambda)]$ is said to be λ-cut of $\tilde{A}$.

Definition 6: Let $\tilde{A}_1 = (m_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, \alpha_2, \beta_2)_{LR}$ be two LR fuzzy numbers and k be a non-zero real number. Then the exact formula for the extended addition and the scalar multiplication are defined as follows:

i) $(m_1, \alpha_1, \beta_1)_{LR} + (m_2, \alpha_2, \beta_2)_{LR} = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$

ii) $k > 0, \; k(m_1, \alpha_1, \beta_1)_{LR} = (km_1, k\alpha_1, k\beta_1)_{LR}$

iii) $k < 0, \; k(m_1, \alpha_1, \beta_1)_{LR} = (km_1, -k\beta_1, -k\alpha_1)_{LR}$

Definition 7: (Extension Principle) This principle allows the generalization of crisp mathematical concepts in fuzzy frameworks. For any function *f*, mapping points in set *X* to points in set *Y*, and any fuzzy set $A \in \tilde{P}(X)$ where $A = \mu_1(x_1) + \mu_2(x_2) + ... + \mu_n(x_n)$, this principle expresses:

$$f(A) = f\left(\mu_1(x_1) + \mu_2(x_2) + \ldots + \mu_n(x_n)\right) = f\left(\mu_1(x_1)\right) + f\left(\mu_2(x_2)\right) + \ldots + f\left(\mu_n(x_n)\right).$$

Definition 8: Let $(\Theta, P(\Theta), Pos)$ be a possibility space where Θ is a non-empty set involving all possible events, and $P(\Theta)$ is the power set of Θ. For every $A \in P(\Theta)$, there is a non-negative number $Pos(A)$, so-called a possibility measure, satisfying the following axioms:

(i) $P(\varnothing) = 0, \quad P(\Theta) = 1$,

(ii) for every $A, B \in P(\Theta)$, $A \subseteq B$ implies $Pos(A) \le Pos(B)$,

(iii) for every subset $\{A_w : w \in W\} \subseteq P(\Theta)$, $Pos(\bigcup_w A_w) = Sup_w Pos(A_w)$.

The elements of $P(\Theta)$ are also called fuzzy events.

Definition 9 (Liu and Liu, [24]): Let ξ be a fuzzy variable on a possibility space $(\Theta, P(\Theta), Pos)$. The possibility, necessity and credibility of the fuzzy event $\{\xi \ge r\}$, where *r* is any real number, is defined as follows:

$$Pos(\xi \ge r) = \underset{t \ge r}{Sup}\, \mu_\xi(t).$$

$$Nes\{\xi \ge r\} = 1 - \underset{t < r}{Sup}\, \mu_\xi(t),$$

$$Cr(\xi \ge r) = \frac{1}{2}\left[Pos\{\xi \ge r\} + Nec\{\xi \ge r\}\right].$$

where $\mu_{\tilde{A}} : R \to [0,1]$ is the membership function of ξ and *r* is a real number. Note here that $Cr(\xi \ge r) = 1 - Cr(\xi < r)$.

Definition 10 (Liu and Liu, [25]): Let (Ω, A, Pr) be a probability space where Ω is a sample space, A is the *s*-algebra of subsets of Ω (i.e., the set of all possible potentially interesting events), and Pr is a probability measure on Ω. A fuzzy random variable (FRV) is a function ξ from a probability space (Ω, A, Pr) to the set of fuzzy variables such that for every Borel set *B* of $\Re$, $Pos\{\xi(w), w \in B\}$ is a measurable function of ω.

Definition 11 (Liu and Liu, [24]): A fuzzy random vector is a map from a sample space to a colleation of fuzzy vectors, $\xi = (\xi_1, \xi_2, ..., \xi_n) : \Omega \to F_v^n$, such that for any closed subset $F \in \Re^n$, $Pos\{\gamma \mid \xi(\omega, \gamma) \in F\}$ is a measurable fuction of $\omega \in \Omega$, i.e., for any $t \in [0,1]$, we have $\left\{\omega \in \Omega \middle| Pos\left\{\gamma \middle| \xi(\omega, \gamma) \in F\right\} \le t\right\} \in A$. In the case of n=1, ξ is called a fuzzy random variable.

Definition 12 (Fuzzy Random Arithmetic): Let ξ_1 and ξ_2 be two FRVs with the probability spaces (Ω_1, A_1, Pr_1) and (Ω_2, A_2, Pr_2), respectively. Then $\xi = \xi_2 + \xi_2$ is defined as $\xi(\omega_1, \omega_2) = \xi_1(\omega_1) + \xi_2(\omega_2)$ for any $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, where $(\Omega_1 \times \Omega_2, A_1 \times A_2, Pr_1 \times Pr_2)$ is the corresponding probability space.

Definition 13: Let $\xi = (\xi_1, \xi_2, ..., \xi_n)$ be a fuzzy random vector, and $f : \Re^n \to \Re$ be a continuous function. Then $f(\xi)$ will be a fuzzy random variable.

Definition 14: An *LR* fuzzy random variable will be denoted by $\xi(\omega)$, where $\omega \in \Omega$ and described by the following membership function:

$$\mu_{\xi(\omega)}(x) = \begin{cases} L\left(\dfrac{m(\omega) - x}{\alpha}\right), & x \le m(\omega), \\ 1 & x = m(\omega), \\ R\left(\dfrac{x - m(\omega)}{\beta}\right), & x \ge m(\omega). \end{cases}$$

where $m(\omega)$ is the normally distributed random variable.

3. Conventional DEA-UO model with crisp data

Consider the performance of a homogeneous set of n DMUs $(DMU_j; j=1,...,n)$ with multiple inputs and multiple outputs is to be evaluated. A production process with m inputs $x_{ij}\,(i=1,...,m)$ to yield s outputs in which s_1 outputs are desirable $y^g_{rk}\,(r=1,...,s_1)$ and s_2 outputs are undesirable $y^b_{pk}\,(p=1,...,s_2)$ such that $s=s_1+s_2$.Several approaches have been developed to deal with undesirable outputs in DEA model. Among these, we preferred Puri and Yadav (2014) approach to suggest a modified version of Korhonen and Luptacik (2004)'s DEA-UO model such that the efficiency ratio for any DMU is guaranteed to be positive. Their proposed model is as follows:

$$
\begin{aligned}
&\max\ \phi\\
&\phi \le \sum_{r=1}^{s_1} u^g_r y^g_{rk} - \sum_{p=1}^{s_2} u^b_p y^b_{pk}\\
&\sum_{i=1}^{m} v_i x_{ik} = 1\\
&\sum_{r=1}^{s_1} u^g_r y^g_{rj} - \sum_{p=1}^{s_2} u^b_p y^b_{pj} \ge 0,\ \forall j\\
&\sum_{r=1}^{s_1} u^g_r y^g_{rk} - \sum_{p=1}^{s_2} u^b_p y^b_{pk} - \sum_{i=1}^{m} v_i x_{ik} \le 0,\ \forall j\\
&u^g_r \ge 0 \forall r, u^b_p \ge 0\,\forall p, v_i \ge 0\,\forall i.
\end{aligned}
\tag{2}
$$

where x_{ik}, y^g_{rk}, and y^b_{pk} are the inputs, desirable outputs and undesirable outputs of target DMU_k, respectively.

Obviously, from model (2), it can be seen that $\phi \le 1$. Hence, we add this constraint to model (2) to get a equivalence model (3) as follows:

$$
\begin{aligned}
&\max\ \phi\\
&\phi \le 1\\
&\phi \le \sum_{r=1}^{s_1} u^g_r y^g_{rk} - \sum_{p=1}^{s_2} u^b_p y^b_{pk}\\
&\sum_{i=1}^{m} v_i x_{ik} = 1\\
&\sum_{r=1}^{s_1} u^g_r y^g_{rj} - \sum_{p=1}^{s_2} u^b_p y^b_{pj} \ge 0,\quad \forall j\\
&\sum_{r=1}^{s_1} u^g_r y^g_{rk} - \sum_{p=1}^{s_2} u^b_p y^b_{pk} - \sum_{i=1}^{m} v_i x_{ik} \le 0,\ \forall j\\
&u^g_r \ge 0 \forall r, u^b_p \ge 0\,\forall p, v_i \ge 0\,\forall i
\end{aligned}
\tag{3}
$$

The main aim of construction model (3) is to enhance the ability of model (2) in uncertain environment to preserve the DEA structure. The reason of this modification is to keep the efficiency values of original model (2) in fuzzy stochastic environment in the range of zero and one when it is transformed into a deterministic one.

This is ignorable, in real life problems, which certain mathematics is not sufficient to model a complex system. In DEA system, input and output parameters may be faced with fuzziness and randomness together. In the present study, to deal with such situations, we extend the DEA-UO model to the fuzzy stochastic DEA (FSDEA) model with undesirable fuzzy stochastic outputs (FSDEA-UFSO).

4. Fuzzy Stochastic DEA-UO model: A probability- possibility approach

Consider n DMUs, indexed by $j=1,2,\ldots,n$ each of which consumes m fuzzy random inputs, denoted by $\tilde{\tilde{x}}_{ij}=(\tilde{x}_{ij},x_{ij}^{\alpha},x_{ij}^{\beta})_{LR}, i=1,...,m$ to produce s_1+s_2 fuzzy random outputs, denoted by $\tilde{\tilde{y}}_{rj}^{g}=(\tilde{y}_{rj}^{g},y_{rj}^{g,\alpha},y_{rj}^{g,\beta})_{LR}, r=1,...,s_1$ as desirable outputs and $\tilde{\tilde{y}}_{pj}^{b}=(\tilde{y}_{pj}^{b},y_{pj}^{b,\alpha},y_{pj}^{b,\beta})_{LR}, p=1,...,s_2$ as undesirable outputs. Let the random parameters $\tilde{x}_{ij}, \tilde{y}_{rj}^{g}, \tilde{y}_{pj}^{b}$ be normally distributed as $N(x_{ij},\sigma_{ij})$, $N(y_{rj}^{g},\sigma_{rj}^{g})$, $N(y_{pj}^{b},\sigma_{pj}^{b})$, respectively, where $x_{ij}, y_{rj}^{g}, y_{pj}^{b}$ and $\sigma_{ij}, \sigma_{rj}^{g}, \sigma_{pj}^{b}$ are the mean value and the variance for $\tilde{x}_{ij}$, $\tilde{y}_{rj}^{g}, \tilde{y}_{pj}^{b}$, respectively.

The chance-constrained programming (CCP) developed by Cooper et al. [4] is a stochastic optimization approach suitable for solving optimization problems with uncertain parameters. Building on CCP and possibility theory as the principal techniques, the following probability-possibility CCR model is proposed:

$$
\begin{aligned}
&\max \ \varphi \\
&s.t. \\
&\phi \le 1 \\
&\left.\begin{aligned}
&\varphi \le \sum_{r=1}^{s_1}\hat{y}_{rk}^{g}-\sum_{p=1}^{s_2}\hat{y}_{pk}^{b} \\
&\sum_{i=1}^{m}\hat{x}_{ij}=1 \\
&\sum_{r=1}^{s_1}\hat{y}_{rj}^{g}-\sum_{p=1}^{s_2}\hat{y}_{pj}^{b}-\sum_{i=1}^{m}\hat{x}_{ij}\le 0 \quad \forall \mathrm{j} \\
&\sum_{r=1}^{s_1}\hat{y}_{rj}^{g}-\sum_{p=1}^{s_2}\hat{y}_{pj}^{b}\le 0 \quad \forall \mathrm{j}
\end{aligned}\right\} \qquad (i) \qquad\qquad (4)\\
&\max \ \varphi \\
&Pr\left[Pos\left(u_r^{g}\tilde{\tilde{y}}_{rj}^{g}\le \hat{y}_{rj}^{g}\le u_r^{g}\tilde{\tilde{y}}_{rj}^{g}\right)\ge \delta\right]\ge \gamma, \quad \forall r, \mathrm{j}, \qquad (ii)\\
&Pr\left[Pos\left(u_p^{b}\tilde{\tilde{y}}_{pj}^{b}\le \hat{y}_{pj}^{b}\le u_p^{b}\tilde{\tilde{y}}_{pj}^{b}\right)\ge \delta\right]\ge \gamma, \quad \forall p, \mathrm{j}, \qquad (iii)\\
&Pr\left[Pos\left(v_i\tilde{\tilde{x}}_{ij}\le \hat{x}_{ij}\le v_i\tilde{\tilde{x}}_{ij}\right)\ge \delta\right]\ge \gamma, \quad \forall i, j, \qquad (iv)\\
&u_r^{g}, u_p^{b}, v_i \ge 0.
\end{aligned}
$$

where δ and $\gamma \in [0,1]$ in constraint (ii), (iii) and (iv) are the predetermined thresholds defined by the DM. $Pos[\cdot]$ and $Pr[\cdot]$ in Model (4) denote the possibility and the probability of $[\cdot]$ event.

In addition, we presume that the fuzzy stochastic input $\tilde{\tilde{x}}_{ij}$ and the fuzzy stochastic output $\tilde{\tilde{y}}_{rj}^{g}$ and $\tilde{\tilde{y}}_{pj}^{b}$ are characterized, respectively, by the following two membership functions:

$$\mu_{\tilde{\bar{x}}_{ij}}(t)=\begin{cases} L\left(\dfrac{\tilde{x}_{ij}-t}{x_{ij}^{\alpha}}\right), & t\le \tilde{x}_{ij} \\ R\left(\dfrac{t-\tilde{x}_{ij}}{x_{ij}^{\beta}}\right), & t\ge \tilde{x}_{ij} \end{cases} \tag{5}$$

and

$$\mu_{\tilde{\bar{y}}_{rj}^{g}}(t)=\begin{cases} L\left(\dfrac{\tilde{y}_{rj}^{g}-t}{y_{rj}^{g,\alpha}}\right), & t\le \tilde{y}_{rj}^{g} \\ R\left(\dfrac{t-\tilde{y}_{rj}^{g}}{y_{rj}^{g,\beta}}\right), & t\ge \tilde{y}_{rj}^{g} \end{cases}, \quad \mu_{\tilde{\bar{y}}_{rj}^{b}}(t)=\begin{cases} L\left(\dfrac{\tilde{y}_{rj}^{b}-t}{y_{rj}^{b,\alpha}}\right), & t\le \tilde{y}_{rj}^{b} \\ R\left(\dfrac{t-\tilde{y}_{rj}^{b}}{y_{rj}^{b,\beta}}\right), & t\ge \tilde{y}_{rj}^{b} \end{cases} \tag{6}$$

where $\tilde{x}_{ij}\sim N(x_{ij},\sigma_{ij}^2)$ and $\tilde{y}_{rj}^{g}\sim N(y_{rj}^{g},\sigma_{rj}^2), \tilde{y}_{pj}^{b}\sim N(y_{pj}^{b},\sigma_{pj}^2)$.

In order to solve the probability-possibility constrained programming Model (4), we convert the constraints in this model into their respective crisp equivalents. Thereby, Theorem 1 and Lemma 1 proposed, respectively, by Liu and Liu [25] and Sakawa [35] play a pivotal role in solving the fuzziness of proposed Model (4).

Theorem 1: Let ξ be a fuzzy random vector $g_j:\Re^n\to\Re$ are real-valued continuous functions $r=1,\dots,p$. Then the possibility $Pos\left\{g_j\left(\xi(w)\right)\le 0,\ j=1,\dots,n\right\}$ is a random variable.

Lemma 1: Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be two independent fuzzy numbers with continuous membership functions. For a given confidence level $\alpha\in[0,1]$, $Pos\left\{\bar{\lambda}_1\ge\bar{\lambda}_2\right\}\ge\alpha$ if and only if $\lambda_{1,\alpha}^{R}\ge\lambda_{2,\alpha}^{R}$, where $\lambda_{1,\alpha}^{L}$, $\lambda_{1,\alpha}^{R}$ and $\lambda_{2,\alpha}^{L}$, $\lambda_{2,\alpha}^{R}$ are the left and the right side extreme points of the α-level sets $\bar{\lambda}_1$ and $\bar{\lambda}_2$, respectively, and $Pos\{\bar{\lambda}_1\ge\bar{\lambda}_2\}$ present the degree of possibility.

In what follows we show that the probability-possibility CCR model (4) can be equivalently transformed into a linear programing model.

The constraint (*ii*) in Model (4), $Pr\left[Pos\left(u_r\tilde{\bar{y}}_{rj}\le\hat{y}_{rj}\le u_r\tilde{\bar{y}}_{rj}\right)\ge\delta\right]\ge\gamma$, can be transformed into the following two constraints:

$$Pr\left[Pos\left(u_r\tilde{\bar{y}}_{rj}\le\hat{y}_{rj}\right)\ge\delta\right]\ge\gamma,$$
$$Pr\left[Pos\left(\hat{y}_{rj}\le u_r\tilde{\bar{y}}_{rj}\right)\ge\delta\right]\ge\gamma.$$

These constraints can be rewritten as the following constraints based on Lemma 1:

$$Pr\left[Pos\left(\tilde{\bar{y}}_{rj}\le\frac{\hat{y}_{rj}}{u_r}\right)\ge\delta\right]\ge\gamma\Leftrightarrow Pr\left(\left(\tilde{\bar{y}}_{rj}\right)_{\delta}^{L}\le\frac{\hat{y}_{rj}}{u_r}\right)\ge\gamma\Leftrightarrow Pr\left(\tilde{y}_{rj}-L^{-1}(\delta)y_{rj}^{\alpha}\le\frac{\hat{y}_{rj}}{u_r}\right)\ge\gamma$$
$$Pr\left[Pos\left(\frac{\hat{y}_{rj}}{u_r}\le\tilde{\bar{y}}_{rj}\right)\ge\delta\right]\ge\gamma\Leftrightarrow Pr\left(\frac{\hat{y}_{rj}}{u_r}\le\left(\tilde{\bar{y}}_{rj}\right)_{\delta}^{R}\right)\ge\gamma\Leftrightarrow Pr\left(\frac{\hat{y}_{rj}}{u_r}\le\tilde{y}_{rj}+R^{-1}(\delta)y_{rj}^{\beta}\right)\ge\gamma$$

In a similar way, constraint (*iii*) in Model (4), $Pr\left[Pos\left(v_i x_{ij}\le\hat{x}_{ij}\le v_i x_{ij}\right)\ge\delta\right]\ge\gamma$, can be rewritten as the following constraints:

$$Pr\left[Pos\left(\tilde{\bar{x}}_{ij}\le\frac{\hat{x}_{ij}}{v_i}\right)\ge\delta\right]\ge\gamma\Leftrightarrow Pr\left(\left(\tilde{\bar{x}}_{ij}\right)_{\delta}^{L}\le\frac{\hat{x}_{ij}}{v_i}\right)\ge\gamma\Leftrightarrow Pr\left(\tilde{x}_{ij}-L^{-1}(\delta)x_{rj}^{\alpha}\le\frac{\hat{x}_{ij}}{v_i}\right)\ge\gamma$$
$$Pr\left[Pos\left(\frac{\hat{x}_{ij}}{v_i}\le\tilde{\bar{x}}_{ij}\right)\ge\delta\right]\ge\gamma\Leftrightarrow Pr\left(\frac{\hat{x}_{ij}}{v_i}\le\left(\tilde{x}_{ij}\right)_{\delta}^{R}\right)\ge\gamma\Leftrightarrow Pr\left(\frac{\hat{x}_{ij}}{v_i}\le\tilde{x}_{ij}+R^{-1}(\delta)x_{ij}^{\beta}\right)\ge\gamma$$

Therefore, Model (4) can be reformulated as follows:

$$
\begin{aligned}
&\max\ \varphi \\
&s.t. \\
&\phi \le 1 \\
&\varphi \le \sum_{r=1}^{s_1} \hat{y}^{g}_{rk} - \sum_{p=1}^{s_2} \hat{y}^{b}_{pk} \\
&\sum_{i=1}^{m} \hat{x}_{ij} = 1 \\
&\sum_{r=1}^{s_1} \hat{y}^{g}_{rj} - \sum_{p=1}^{s_2} \hat{y}^{b}_{pj} - \sum_{i=1}^{m} \hat{x}_{ij} \le 0 \quad \forall \mathrm{j} \\
&\sum_{r=1}^{s_1} \hat{y}^{g}_{rj} - \sum_{p=1}^{s_2} \hat{y}^{b}_{pj} \le 0 \quad \forall \mathrm{j} \\
&Pr\left(\tilde{y}^{g}_{rj} - L^{-1}(\delta) y^{\alpha,g}_{rj} \le \frac{\hat{y}^{g}_{rj}}{u^{g}_{r}} \le \tilde{y}^{g}_{rj} + R^{-1}(\delta) y^{\beta,g}_{rj} \right) \ge \gamma, \quad \forall r, j \qquad (i) \\
&Pr\left(\tilde{y}^{b}_{pj} - L^{-1}(\delta) y^{\alpha,b}_{pj} \le \frac{\hat{y}^{b}_{pj}}{u^{b}_{p}} \le \tilde{y}^{b}_{pj} + R^{-1}(\delta) y^{\beta,b}_{pj} \right) \ge \gamma, \quad \forall p, j \qquad (ii) \\
&Pr\left(\tilde{x}_{ij} - L^{-1}(\delta) x^{\alpha}_{rj} \le \frac{\hat{x}_{ij}}{v_i} \le \tilde{x}_{ij} + R^{-1}(\delta) x^{\beta}_{ij} \right) \ge \gamma, \quad \forall i, j \qquad (iii) \\
&u^{g}_{r}, u^{b}_{r}, v_i \ge 0.
\end{aligned} \tag{7}
$$

By the help of standardized normal distribution, (see, e.g., Cooper et al. [4]), Model (7) can be transformed into a deterministic linear programming model. Consequently, let us consider the first inequality in constraint (*i*) of Model (7) as $Pr\left(\tilde{h} \ge 0\right) \ge \gamma$ where $\tilde{h} = \tilde{y}^{g}_{rj} - L^{-1}(\delta) y^{\alpha,g}_{rj} - \frac{\hat{y}^{g}_{rj}}{u^{g}_{r}}$. Due to the normal distribution of $y^{m,g}_{rj}$, $\tilde{h}$ also has normal distribution with the following mean and variance:

$$
E(\tilde{h}) = E\left[\tilde{y}^{g}_{rj} + R^{-1}(\delta) y^{\beta,g}_{rj} - \frac{\hat{y}^{g}_{rj}}{u^{g}_{r}} \right] = y^{g}_{rj} + R^{-1}(\delta) y^{\beta,g}_{rj} - \frac{\hat{y}^{g}_{rj}}{u^{g}_{r}}
$$

$$
Var(\tilde{h}) = Var\left(y^{g}_{rj} + R^{-1}(\delta) y^{\beta,g}_{rj} - \frac{\hat{y}^{g}_{rj}}{u^{g}_{r}} \right) = Var\left(\tilde{y}^{g}_{rj}\right) = \sigma^{2}_{rj}
$$

By standardizing the normal distribution, $Pr\left(\tilde{h} \ge 0\right) \ge \gamma$ is converted to

$$
Pr\left(z \ge \frac{-E\left(\tilde{h}\right)}{\sqrt{\mathrm{var}\left(\tilde{h}\right)}} \right) \ge \gamma
$$

where $z = \frac{h - E\left(\tilde{h}\right)}{\sqrt{\mathrm{var}\left(\tilde{h}\right)}}$ is the standard normal random variable with zero mean and unit variance. The corresponding cumulative distribution function is

$$
\Phi\left(\frac{-E(\tilde{h})}{\sqrt{Var(\tilde{h})}} \right) \le 1 - \gamma
$$

and it is equal to $\frac{\hat{y}_{rj}^{g}}{u_r^{g}} - y_{rj}^{g} - R^{-1}(\delta) y_{rj}^{\beta,g} \le \sigma_{rj}\Phi_{1-\gamma}^{-1}$, where $\Phi_{1-\gamma}^{-1}$ is the inverse of Φ at the level of $1-\gamma$. Finally, the deterministic version of constraint (i) in Model (7) will be as follows:

$$\hat{y}_{rj}^{g} \le u_r^{g}(y_{rj}^{g} + R^{-1}(\delta) y_{rj}^{\beta,g} + \sigma_{rj}\Phi_{1-\gamma}^{-1}),\ \forall r, j$$

A similar procedure adopted for constraints (*ii*), (*iii*) and (*iv*) in Model (7) results in the following constraints:

$$\begin{aligned}
(i):\ & \hat{y}_{rj}^{g} \le u_r^{g}(y_{rj}^{g} + R^{-1}(\delta) y_{rj}^{\beta,g} + \sigma_{rj}\Phi_{1-\gamma}^{-1}),\ \forall r, j \\
& u_r^{g}(y_{rj}^{g} - L^{-1}(\delta) y_{rj}^{\alpha,g} - \sigma_{rj}\Phi_{1-\gamma}^{-1}) \le \hat{y}_{rj}^{g},\ \forall r, j
\end{aligned}$$

$$\begin{aligned}
(ii):\ & u_p^{b}(y_{pj}^{b} - L^{-1}(\delta) y_{pj}^{\alpha,b} - \sigma_{pj}\Phi_{1-\gamma}^{-1}) \le \hat{y}_{pj}^{b},\ \forall p, j \\
& u_p^{b}(y_{pj}^{b} - L^{-1}(\delta) y_{pj}^{\alpha,b} - \sigma_{pj}\Phi_{1-\gamma}^{-1}) \le \hat{y}_{pj}^{b},\ \forall p, j
\end{aligned}$$

$$\begin{aligned}
(iii):\ & \hat{x}_{ij} \le v_i(x_{ij} + R^{-1}(\delta) x_{ij}^{\beta} + \sigma_{ij}\Phi_{1-\gamma}^{-1}),\ \forall i, j \\
& v_i(x_{ij} - L^{-1}(\delta) x_{ij}^{\alpha} - \sigma_{ij}\Phi_{1-\gamma}^{-1}) \le \hat{x}_{ij},\ \forall i, j
\end{aligned}$$

As a consequence, the deterministic equivalent for Model (4) can be set as follows:

$$\begin{aligned}
& E_k(\delta,\gamma) = \max\ \varphi \\
& s.t. \\
& \phi \le 1 \\
& \varphi \le \sum_{r=1}^{s_1} \hat{y}_{rk}^{g} - \sum_{p=1}^{s_2} \hat{y}_{pk}^{b} \\
& \sum_{i=1}^{m} \hat{x}_{ij} = 1 \\
& \sum_{r=1}^{s_1} \hat{y}_{rj}^{g} - \sum_{p=1}^{s_2} \hat{y}_{pj}^{b} - \sum_{i=1}^{m} \hat{x}_{ij} \le 0 \quad \forall \mathrm{j} \\
& \sum_{r=1}^{s_1} \hat{y}_{rj}^{g} - \sum_{p=1}^{s_2} \hat{y}_{pj}^{b} \le 0 \quad \forall \mathrm{j} \\
& u_r^{g}(y_{rj}^{g} - L^{-1}(\delta) y_{rj}^{\alpha,g} - \sigma_{rj}\Phi_{1-\gamma}^{-1}) \le \hat{y}_{rj}^{g} \le u_r^{g}(y_{rj}^{g} + R^{-1}(\delta) y_{rj}^{\beta,g} + \sigma_{rj}\Phi_{1-\gamma}^{-1}), \quad \forall r, j \\
& u_r^{b}(y_{rj}^{b} - L^{-1}(\delta) y_{pj}^{\alpha,b} - \sigma_{pj}\Phi_{1-\gamma}^{-1}) \le \hat{y}_{pj}^{b} \le u_p^{b}(y_{pj}^{b} + R^{-1}(\delta) y_{pj}^{\beta,b} + \sigma_{pj}\Phi_{1-\gamma}^{-1}), \quad \forall p, j \\
& v_i(x_{ij} - L^{-1}(\delta) x_{ij}^{\alpha} - \sigma_{ij}\Phi_{1-\gamma}^{-1}) \le \hat{x}_{ij} \le v_i(x_{ij} + R^{-1}(\delta) x_{ij}^{\beta} + \sigma_{ij}\Phi_{1-\gamma}^{-1}), \qquad \forall i, j \\
& u_r^{g}, u_p^{b}, v_i \ge 0.
\end{aligned} \tag{8}$$

The above model is obviously a linear program. It should be noted that the deterministic model obtained by Tavana et al. [38] is a non-linear program.

The following theorem shows that the objective function of Model (8), $E_k(\delta,\gamma)$, is monotonously decreasing related to the each of δ and γ level.

Theorem 2: If $E_k(\delta,\gamma)$ is the optimum objective function value of Model (8) then $E_k(\delta_1,\gamma) \ge E_k(\delta_2,\gamma)$ and $E_k(\delta,\gamma_1) \ge E_k(\delta,\gamma_2)$ where $\delta_1 \le \delta_2$ and $\gamma_1 \le \gamma_2$.

Proof. Denote the feasible space of Model (8) by $S_{\delta,\gamma}$. We need to prove that $S_{\delta_2,\gamma_2} \subseteq S_{\delta_1,\gamma_1}$.To this, consider the following constraint of Model (8)

$$v_i(x_{ij} - L^{-1}(\delta)x^{\alpha}_{ij} - \sigma_{ij}\Phi^{-1}_{1-\gamma}) \le \hat{x}_{ij} \le v_i(x_{ij} + R^{-1}(\delta)x^{\beta}_{ij} + \sigma_{ij}\Phi^{-1}_{1-\gamma}) \quad (9)$$

Let $\Phi^{-1}(\gamma) = \Phi^{-1}_{\gamma}$. As $\Phi^{-1}(1-\gamma)$, $L^{-1}(\delta)$ and $R^{-1}(\delta)$ are decreasing function, the functions $-\Phi^{-1}(1-\gamma), -L^{-1}(\delta)$ and $-R^{-1}(\delta)$ will be increasing. It is concluded that

$$\left[x_{ij} - L^{-1}(\delta_2)x^{\alpha}_{ij} - \sigma_{ij}\Phi^{-1}(1-\gamma_2), x_{ij} + R^{-1}(\delta_2)x^{\beta}_{ij} + \sigma_{ij}\Phi^{-1}(1-\gamma_2)\right] \subseteq \left[x_{ij} - L^{-1}(\delta_1)x^{\alpha}_{ij} - \sigma_{ij}\Phi^{-1}(1-\gamma_1), x_{ij} + R^{-1}(\delta_1)x^{\beta}_{ij} + \sigma_{ij}\Phi^{-1}(1-\gamma_1)\right]$$

This completes the proof.☐

We present the following defiition to define the efficiecy of each DMU.

Definition 16.For the given level δ and γ, we define $E^T_k(\delta,\gamma) = E_k(\delta,\frac{\gamma}{2})$ as efficiency score of DMU$_k$ in fuzzy random DEA model (8).

The corresponding model with $E^T_k(\delta,\gamma)$ is as follows:

$$\begin{aligned}
& E^T_k(\delta,\gamma) = \max\ \varphi \\
& s.t. \\
& \phi \le 1 \\
& \varphi \le \sum_{r=1}^{s_1} \hat{y}^g_{rk} - \sum_{p=1}^{s_2} \hat{y}^b_{pk} \\
& \sum_{i=1}^{m} \hat{x}_{ij} = 1 \\
& \sum_{r=1}^{s_1} \hat{y}^g_{rj} - \sum_{p=1}^{s_2} \hat{y}^b_{pj} - \sum_{i=1}^{m} \hat{x}_{ij} \le 0 \quad \forall j \\
& \sum_{r=1}^{s_1} \hat{y}^g_{rj} - \sum_{p=1}^{s_2} \hat{y}^b_{pj} \le 0 \quad \forall j \\
& u^g_r(y^g_{rj} - L^{-1}(\delta)y^{\alpha,g}_{rj} - \sigma_{rj}\Phi^{-1}_{1-\frac{\gamma}{2}}) \le \hat{y}^g_{rj} \le u^g_r(y^g_{rj} + R^{-1}(\delta)y^{\beta,g}_{rj} + \sigma_{rj}\Phi^{-1}_{1-\frac{\gamma}{2}}), \quad \forall r, j \\
& u^b_r(y^b_{rj} - L^{-1}(\delta)y^{\alpha,b}_{pj} - \sigma_{pj}\Phi^{-1}_{1-\frac{\gamma}{2}}) \le \hat{y}^b_{pj} \le u^b_p(y^b_{pj} + R^{-1}(\delta)y^{\beta,b}_{pj} + \sigma_{pj}\Phi^{-1}_{1-\frac{\gamma}{2}}), \quad \forall p, j \\
& v_i(x_{ij} - L^{-1}(\delta)x^{\alpha}_{ij} - \sigma_{ij}\Phi^{-1}_{1-\frac{\gamma}{2}}) \le \hat{x}_{ij} \le v_i(x_{ij} + R^{-1}(\delta)x^{\beta}_{ij} + \sigma_{ij}\Phi^{-1}_{1-\frac{\gamma}{2}}), \quad \forall i, j \\
& u^g_r, u^b_p, v_i \ge 0.
\end{aligned} \quad (10)$$

Theorem 3: Consider $E^{\mathrm{T}}_k(\delta,\gamma)$ as the optimum objective function value of Model (10) for DMU$_k$, then

(a). $E^{\mathrm{T}}_k(\delta_1,\gamma) \ge E^T_k(\delta_2,\gamma)$ and $E^T_k(\delta,\gamma_1) \ge E^T_k(\delta,\gamma_2)$ where $\delta_1 \le \delta_2$ and $\gamma_1 \le \gamma_2$.

(b). $0 < E^T_j(\delta,\gamma) \le 1, (j = 1,2,\ldots,n)$. Also, theres exists at lease one $k \in \{1,2,\ldots,n\}$ suhc that $E^T_k(\delta,\gamma) = 1$.

(c). Model (10) is feasible for any δ and γ.

Proof: (a). It is straightforward using Theorem 2 and Definition 16.

(b). Obviously, it is followed immediately from the first, second and third constraints of Model (9) that $E^T_j(\delta,\gamma) \le 1$. In what follows, we introduce such DMU$_k$ with $E^{\mathrm{T}}_k(\delta,\gamma) = 1$. According to part (a), $E^{\mathrm{T}}_k(\delta,\gamma)$ is decreasing with respect to both δ and γ threshold, and so $E^{\mathrm{T}}_k(\delta,\gamma) \ge E^{\mathrm{T}}_k(1,1)$. Let $\delta = 1$ and $\gamma = 1$, then

$L^{-1}(1) = R^{-1}(1) = 0$ and $\Phi^{-1}(0.5) = 0$. Hence, we have $\hat{x}_{ij} = v_i x_{ij}, \hat{y}_{rj} = u_r y_{rj}$ in Model (10). Therefore, The coorespondig model with $E_k^{\mathrm{T}}(1,1)$ will be as follows:

$$
\begin{aligned}
& E_k^{\mathrm{T}}(1,1) = \phi \\
& s.t. \\
& \phi \le 1 \\
& \phi \le \sum_{r=1}^{s_1} u_r^g y_{rk}^g - \sum_{p=1}^{s_2} u_p^b y_{pk}^b \\
& \sum_{i=1}^{m} v_i x_{ik} = 1 \qquad (11) \\
& \sum_{r=1}^{s_1} u_r^g y_{rj}^g - \sum_{p=1}^{s_2} u_p^b y_{pj}^b \ge 0, \quad \forall j \\
& \sum_{r=1}^{s_1} u_r^g y_{rk}^g - \sum_{p=1}^{s_2} u_p^b y_{pk}^b - \sum_{i=1}^{m} v_i x_{ik} \le 0, \ \forall j \\
& u_r^g \ge 0 \forall r, u_p^b \ge 0 \forall p, v_i \ge 0 \forall i
\end{aligned}
$$

As seen the above model is same with DEA-UO model given in (1)), and this model corresponds to a standard DEA model when the feasible solution $u_p^b = 0, \forall p$ are considered. So, $E_k^{\mathrm{T}}(1,1)$ would be positve as the objective function value of a traditional DEA-UO model and then $E_k^{\mathrm{T}}(\delta,\gamma) \ge E_k^{\mathrm{T}}(1,1) > 0$. On the other hand, for such DMU_k, $E_k^{\mathrm{T}}(1,1)$ would be equal to 1. Hence, the relation $1 \ge E_k^{\mathrm{T}}(\delta,\gamma) \ge E_k^{\mathrm{T}}(1,1) = 1$ completes the proof of part (b).

(c). Denote the feasible space of Model (10) by $S_{\delta,\gamma}^T$. According to the proof of Theorem 2, $S_{1,1}^T \subseteq S_{\delta,\gamma}^T$. Therefore, it is sufficient to show that the feasible space $S_{1,1}^T$ is nonempty. According to the proof of part (b), $E_k^{\mathrm{T}}(1,1)$ is given by Model (11) and this model is alwasy feasible. This complets the proof of part (c).☐

5. Numerical Example

In this section, a numerical example is considered to get a deep insight of the proposed methodology in (10). Table 1 shows an assessment problem with 20 DMUs in terms of two inputs, two desirable outputs and one undesirable output in fuzzy random environment. The entire input and output data are in terms of symmetric triangular fuzzy random numbers. Each input and output data is denoted by $(N(m,\sigma),\alpha)$, where m is the mean of the random center value in normal distribution and α is the left and also the right tail.

Table 2 shows the evaluating results by Model (10) when we set the predetermined minimum probability level δ and the predetermined acceptable level of possibility γ in five different threshold levels of (δ = 0.25, γ = 0.25), (δ = 0.5, γ = 0.25), (δ = 0.75, γ = 0.25), (δ = 0.5, γ = 0.5), and (δ = 0.25, γ = 0.75). With the variation in the satisfaction levels δ and γ the efficient DMUs are almost DMU1, DMU6, DMU9, DMU10, DMU15, DMU16, and DMU18. Generally from Table 2, we can see the applicability of theorem 3 when the efficiency scores of the DMUs decrease by increasing the level δ from(δ =0.25,γ=0.25) to (δ =0.75, γ=0.25)and increase by decreasing of the level γ from(δ =0.25, γ=0.75) to(δ =0.25, γ=0.25).

Table 3 presents the ACE, i.e. $\bar{E}_k^*(\delta,\gamma)$ for each efficient DMU at levels stated above. Also, these ACE scores are used to obtain a complete ranking of DMUs which is shown in Table 4.

As seen, the complete ranking of DMUs is similar except for some relocation in surrounding DMUs. Another point obtained from Table 2 is the influence of the variations of stochastic level δ that is more than fuzzy level γ on $E_k(\delta,\gamma)$ in this example. Indeed, with the same increasing in each of levels δ and γ the objective value decreases further by increasing the fuzzy level γ, and so the number of efficient DMUs is fall down in this case.

Table 1: The fuzzy random input and output data

DMU	Input 1	Input 2	Desirable Output 1	Desirable Output 2	Undesirable Output
1	(N(363.4,1),28.3,28.3)	(N(39.6,1),2.4,2.4)	(N(81,1),7,7)	(N(230,1),23.5,23.5)	(N(58.3,1),5.3,5.3)
2	(N(586.5,1),53.2,53.2)	(N(99,1), 9, 9)	(N(45,1),3,3)	(N(345,1),36.5,36.5)	(N(53,1),4.6,4.6)
3	(N(540.5,1),48,48)	(N(59.4,1),4.6,4.6)	(N(49,1),3,3)	(N(368,1),38.8,38.8)	(N(79.5,1),7.9,7.9)
4	(N(473.8,1),40,40)	(N(55.8,1),5.2,5.2)	(N(64,1),5,5)	(N(414,1),44,44)	(N(68.9,1),6.6,6.6)
5	(N(561.2,1),50.3,50.3)	(N(54,1),4,4)	(N(59,1),4,4)	(N(216.2,1),22,22)	(N(48.4,1),4,4)
6	(N(616.4,1) ,56.4,56.4)	(N(90,1) ,8,8)	(N(81,1),7,7)	(N(529,1),56.7,56.7)	(N(31.7,1),1.9,1.9)
7	(N(402,1) ,38.7,38.7)	(N(42.3,1) ,3.6,3.6)	(N(41,1),3,3)	(N(295,1),31.6,31.6)	(N(72.4,1),6.8,6.8)
8	(N(653.2,1) ,60.5,60.5)	(N(68.4,1) ,5.6,5.6)	(N(72.4,1),6,6)	(N(349.6,1) ,36.8,36.8)	(N(90.1,1),9.2,9.2)
9	(N(347.3,1) ,26.6,26.6)	(N(36,1) ,2.3,2.3)	(N(90,1),8,8)	(N(437,1) ,46.5,46.5)	(N(100.7,1),10.5,10.5)
10	(N(301.3,1) ,21.4,21.4)	(N(34.2,1) ,1.8,1.8)	(N(99,1),9,9)	(N(549.7,1) ,59,59)	(N(74.2,1),7.2,7.2)
11	(N(523.2,1) ,48.9,48.9)	(N(87.5,1) ,5,5)	(N(67,5,5)	(N(421,1) ,32.3,32.3)	(N(87,1),8.8,8.8)
12	(N(386.4,1) ,31,31)	(N(48.6,1) ,3.4,3.4)	(N(108,1)),10,10)	(N(575,1), 61.8,61.8)	(N(76.2,1),7.5,7.5)
13	(N(785.7,1) ,75.2,75.2)	(N(50.6,1) ,1.4,1.4)	(N(87,1),8,8)	(N(512.9,1) ,54.9,54.9)	(N(111.3,1),11.9,11.9)
14	(N(694.6,1) ,65.1,65.1)	(N(108,1) ,11,11)	(N(78,1),7,7)	(N(471.5,1),50.3,50.3)	(N(95.4,1),9.7,9.7)
15	(N(598,1) ,54.5,54.5)	(N(27,1) ,1.7,1.7)	(N(67,1),6,6)	(N(391,1),41.4,41.4)	(N(42.4,1),3.3,3.3)
16	(N(713,1) ,67.2,67.2)	(N(126,1) ,9,9)	(N(112,1),10,10)	(N(529,1),56.7,56.7)	(N(58.3,1),5.2,5.2)
17	(N(611.8,1),56,56)	(N(97.2,1) ,8.8,8.8)	(N(73,1),6,6)	(N(402.5,1),42.7,42.7)	(N(68.9,1),6.4,6.4)
18	(N(660.1,1),61.3,61.3)	(N(81,1)7,7)	(N(93,1),8,8)	(N(588.8,1),63.4,63.4)	(N(63.6,1),5.9,5.9)
19	(N(529,1),46.7,46.7)	(N(50.4,1),3.6,3.6)	(N(48,1), 3,3)	(N(276,1),28.6,28.6)	(N(105.4,1),11.1,11.1)
20	(N(621,1),57,57)	(N(57.6,1),4.4,4.4)	(N(77,1),7,7)	(N(400.2,1),42.4,42.4)	(N(90.1,1),9.2,9.2)

Table 2: The stochastic fuzzy efficiency scores

DMU	$(\delta=0.25,\gamma=0.25)$	$(\delta=0.5,\gamma=0.25)$	$(\delta=0.75,\gamma=0.25)$	$(\delta=0.5,\gamma=0.5)$	$(\delta=0.25,\gamma=0.75)$
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.5636	0.5469	0.5505	0.4455	0.3536
3	0.8174	0.8138	0.8075	0.7288	0.6586
4	0.7995	0.7593	0.7291	0.5748	0.5271
5	0.7669	0.7008	0.6111	0.5296	0.4513
6	1.0000	1.0000	1.0000	1.0000	1.0000
7	0.5820	0.5617	0.5404	0.5130	0.4873
8	0.4691	0.4517	0.4323	0.4160	0.4007
9	1.0000	1.0000	1.0000	0.9919	0.9683
10	1.0000	1.0000	1.0000	1.0000	1.0000
11	0.5730	0.5708	0.5669	0.5214	0.4800
12	1.0000	1.0000	1.0000	1.0000	1.0000
13	0.7860	0.7616	0.7360	0.7052	0.6761
14	0.4936	0.4920	0.4888	0.4461	0.4075
15	1.0000	1.0000	1.0000	1.0000	1.0000
16	1.0000	1.0000	1.0000	1.0000	1.0000
17	0.5593	0.5334	0.5201	0.4339	0.4006
18	1.0000	1.0000	1.0000	1.0000	0.9058
19	0.4443	0.4302	0.4147	0.3941	0.3746
20	0.5981	0.5752	0.5495	0.5302	0.5116

Table 3: Average Cross Efficiency of efficient DMUs.

DMUs	(δ =0.25, γ=0.25)	(δ =0.5, γ=0.25)	(δ =0.75, γ=0.25)	(δ =0.5, γ=0.5)	(δ =0.25, γ=0.75)
DMU 1	0.9352	0.9216	0.9586	0.8038	0.8702
DMU 6	0.6959	0.6865	0.7763	0.7637	0.7691
DMU 9	0.8398	0.8242	0.9035	-------	-------
DMU 10	1.1361	1.0773	1.3782	1.0435	0.8345
DMU 12	1.3141	1.1669	1.3379	1.2663	1.1546
DMU 15	1.2838	1.2491	1.1606	1.2346	1.0753
DMU 16	0.4981	0.5636	0.6031	0.6155	0.6664
DMU 18	0.7290	0.7019	0.6908	0.6579	-------

Table 4: Complete ranking of the DMUs in different δ and γ

δ	γ	Complete ranking of the DMUs
0.25	0.25	**12>15>10>1>9>18>6>16>**3>4>13>5>20>7>11>2>17>14>8>19
0.50	0.25	**15>12>10>1>9>18>6>16>**3>13>4>5>20>11>7>2>17>14>8>19
0.75	0.25	**12>10>15>1>9>18>6>16>**3>13>4>5>11>2>20>7>17>14>8>19
0.50	0.50	**15>12>10>1>6>18>16>**9>3>13>4>20>5>11>7>14>2>17>8>19
0.25	0.75	**12>15>1>10>6>16>**9>18>13>3>4>20>7>11>5>14>8>17>19>2

6. Conclusions

In this paper, we have developed a DEA model with undesirable output which is extended to fuzzy random environment. we have firstly modified the deterministic DEA-UO model proposed by puri et al [32]. Further, the proposed DEA-UO model is extended to fuzzy random environment. A methodology in chance constraint programming adopted to solve such DEA model. Unlike the proposed model by Tavana et al. [37], our proposed approach not only leads to a linear program, but also it gives efficiency scores with the range of zero to one for DMUs similar to traditional input-oriented DEA models. Also, the proposed model is feasible. For future study, a new measure in fuzzy stochastic programming can also be planned in chance constraint programming.

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