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New Existence Results for Boundary Value Problems with Integral Conditions

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A B S T R A C T

In this paper, we investigate the existence and uniqueness of solution for fractional boundary value problem for nonlinear fractional differential equation with the integral boundary conditions

$$\begin{cases} {}^{ABC}D_t^{\alpha} u(t) = f\left(t, u(t)\right), \quad 0 < t < 1, \\ u(0) - \gamma_1 u(1) = \lambda_1 \int_0^1 g\left(s, u(s)\right) ds, \\ u'(0) - \gamma_2 u'(1) = \lambda_2 \int_0^1 h\left(s, u(s)\right) ds, \end{cases}$$

where ${}^{ABC}D_t^{\alpha}$ denotes Caputo derivative of order α by using the fixed point theory. We apply the contraction mapping principle and Krasnoselskii's fixed point theorem to obtain some new existence and uniqueness results. Two examples are given to illustrate the main results.

1. Introduction

In recent years, fractional calculus is one of the interest issues that attract many scientists, specially mathematics and engineering sciences. Many natural phenomena can be present by boundary value problems of fractional differential equations. Many authors in different field such as chemical physics, fluid flows, electrical networks, viscoelasticity, try to modeling of these phenomena by boundary value problems of fractional differential equations [1, 5]. For achieve extra information in fractional calculus, specially boundary value problems, reader can refer to more valuable papers or books that are written by authors [6, 7, 14, 15]. In boundary value problems, one of the most important factors that cause to write different papers is the variety of boundary condition selection. One of these situations is integral boundary conditions. Integral boundary conditions have various applications in applied

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fields such as underground water flow, blood flow problems and population dynamics (see [16, 18] for more details).

In this paper, we wish to survey the existence findings for the new type nonlinear Langevin equation involving two fractional orders as:

$$\begin{cases} {}_0^{ABC}D_t^{\alpha}u(t) = f\left(t, u(t)\right), \quad 0 < t < 1, \\ u(0) - \gamma_1 u(1) = \lambda_1 \int_0^1 g\left(s, u(s)\right)ds, \\ u'(0) - \gamma_2 u'(1) = \lambda_2 \int_0^1 h\left(s, u(s)\right)ds, \end{cases} \tag{1}$$

where $0 < t < 2$, ${}_0^{ABC}D_t^{\alpha}$ shows the α - th Atanga- Baleanu fractional derivatives and $f : [0,1] \to \Re \times \Re$ is a given continuous function.

Lemma 1: Given $f \in C[0,1]$ and $1 < \alpha \le 2$, the problem (1)-(2), is equivalent to

$$u(t) = N_1 + M_3 t + \int_0^1 G(t,s) f\left(s, u(s)\right)ds + N_6 \int_0^1 g\left(s, u(s)\right)ds + (N_3 + M_5 t)\int_0^1 h\left(s, u(s)\right)ds\ ,$$

where

$$G(t,s) = \begin{cases} N_2(1-s)^{\alpha-2} + N_4(1-s)^{\alpha-2} + N_5(1-s)^{\alpha-1} + M_4 t(1-s)^{\alpha-2} + M_1 + M_4(t-s)^{\alpha-1}\ , & s \le t \\ N_2(1-s)^{\alpha-2} + N_4(1-s)^{\alpha-2} + N_5(1-s)^{\alpha-1} + M_4 t(1-s)^{\alpha-2} & , \quad s \le t \end{cases} \tag{2}$$

$${}^{AB}I^{\alpha}\ {}_0^{ABC}D_t^{\alpha}u(t) = {}^{AB}I^{\alpha} f\left(t, u(t)\right), \quad 0 < t < 1, \tag{3}$$

So, we have

$$u(t) = c_1 + c_2 t + {}^{AB}I^{\alpha} f\left(t, u(t)\right), \quad 0 < t < 1. \tag{4}$$

Letting $\beta = \alpha - 1$,

$$\begin{aligned} ({}^{AB}I^{\alpha} f)(t) = I^1({}^{AB}I^{\beta} f)(t) &= I^1\left\{\frac{1-\beta}{B(\alpha)} f + \frac{\beta}{B(\beta)}(I^{\beta} f)\right\}(t) \\ &= \frac{2-\alpha}{B(\alpha-1)}\int_0^t f(s)ds + \frac{\alpha-1}{B(\alpha-1)} I^{\beta+1} f(t) \\ &= \frac{2-\alpha}{B(\alpha-1)}\int_0^t f(s)ds + \frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds. \end{aligned}$$

Therefore,

$$u(t) = c_1 + c_2 t + \frac{2-\alpha}{B(\alpha-1)}\int_0^t f(s, u(s))ds + \frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, u(s))ds. \tag{5}$$

By using the boundary condition in (1) we obtain

$$c_1 = N_1 + N_2\int_0^1 (1-s)^{\alpha-2} f\,(s,u(s))ds + N_3\int_0^1 h(s,u(s))ds$$

$$+N_4\int_0^1 (1-s)^{\alpha-2} f\,(s,u(s))ds + N_5\int_0^1 (1-s)^{\alpha-1} f\,(s,u(s))ds + N_6\int_0^1 g(s,u(s))ds\,,$$

and

$$c_2 = M_3 + M_4\int_0^1 (1-s)^{\alpha-2} f\,(s,u(s))ds + M_5\int_0^1 h(s,u(s))ds\,,$$

were

$$M_1 = \frac{2-\alpha}{B(\alpha-1)}\,,\ M_2 = \frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha)}\,,\ M_3 = \frac{\gamma_2}{(1-\gamma_2)}M_1 f\,(1,u(1)),$$

$$M_4 = \frac{\gamma_2(\alpha-1)}{(1-\gamma_2)(\alpha-1)\Gamma(\alpha-1)}\,,\ M_5 = \frac{\lambda_2}{1-\gamma_2}\,,$$

and

$$N_1 = \frac{\gamma_1}{(1-\gamma_1)}M_3\,,\ N_2 = \frac{\gamma_1}{(1-\gamma_1)}M_4\,,\ N_3 = \frac{\gamma_1}{(1-\gamma_1)}M_5$$

$$N_4 = \frac{\gamma_1}{(1-\gamma_1)}M_1\,,\ N_5 = \frac{\gamma_1}{(1-\gamma_1)}M_2\,,\ N_6 = \frac{\lambda_1}{1-\gamma_1}.$$

2. Preliminaries

In this section, we study the existence and uniqueness solutions for FBVP (1).

Let $E = \left(C\,[0,1];\mathbb{R}\right)$ be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$. Define operator $T : E \to E$ as

$$Tu(t) = N_1 + M_3 t + \int_0^1 G(t,s) f\,(s,u(s))ds + N_6\int_0^1 g(s,u(s))ds \tag{6}$$

$$+(N_3 + M_5 t)\int_0^1 h(s,u(s))ds\,, \tag{7}$$

Due to the Lemma 1, problem (1) is converted into a fixed point problem $u = Tu$. We behold, the initial problem (1) has solutions if the operator (3) has fixed points.

For convenience of presentation, we now present below hypothesis to be used in the rest of the paper:

(H_1) $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuously differentiable function.

(H_2) There exist $a(t) \in C\left([0,1],[0,\infty)\right)$ such that

i) $\int_0^1 (1-s)^{\alpha-i-1}|\alpha(s)|ds < 1-v_i\,,\ \ i = 1,2\,;$

ii) $\int_0^t (t-s)^{\alpha-1}|\alpha(s)|ds < 1-v_3\,;$

iii) $\int_0^1 |\alpha(s)| ds < 1 - \nu_4$.

(H_3) For $i = 1,2,3$

i) $R_1 = \sup \int_0^1 (1-s)^{\alpha-i-1} |f(s,0)| ds < \infty$;

ii) $R_2 = \sup \int_0^t (t-s)^{\alpha-1} |f(s,0)| ds < \infty$;

iii) $R_3 = \sup_{t\in[0,1]} \int_0^t (t-s)^{\alpha-1} f(s,0)| ds < \infty$.

(H_4) For $u, v \in C[0,\infty]$

i) $|f(t,u(t)) - f(t,u(t))| \le a(t)|u(t) - v(t)|$.

(H_5) There exists positive constants L_1 and L_2, such that

i) $|g(t,u(t)) - g(t,v(t))| \le L_1(1-\nu)|u(t) - v(t)|$.

ii) $|h(t,u(t)) - h(t,v(t))| \le L_2(1-\nu)|u(t) - v(t)|$.

where $u, v \in C[0,\infty]$ and $\nu = \min\{\nu_i;\ i = 1,2,3,4\}$.

(H_6) There exists positive constants $p_i \in \mathrm{C}([0,1],\mathbb{R})$ and $\psi_i \in C([0,1]\times\mathbb{R},\mathbb{R})$, $(i = 1,2,3)$ such that

i) $|f(t,u(t))| \le p_1(t)\psi_1(|u|)$;

ii) $|g(t,u(t))| \le p_2(t)\psi_2(|u|)$;

iii) $|h(t,u(t))| \le p_3(t)\psi_3(|u|)$.

with $\psi_1(|u|) \le \sqrt{|u|}$, $(i = 1,2,3)$.

3. Main results

In this section, we will establish the existence and uniqueness criteria of solutions for the initial problem (1). Then, the existence and uniqueness results for the fixed problem are obtained respectively via Theorem 3 and Theorem 2.

Theorem 1: Assume that $(H_1)-(H_5)$ hold. Let

$$\rho_1 = |M_1| + |M_2| + |M_4| + |N_2| + |N_4| + |N_5| + |N_6|L_1 + (|N_3| + |M_5|)L_2$$

$$\rho_2 = |N_1| + |M_3| + (|N_2| + |N_4| + |M_4|)R_1 + |N_5|R_2 + |M_1|K_1 + |M_2|R_3 + |N_6|K_2 + (|N_3| + |M_5|)K_3$$

$(1-\nu)\rho_1 < 1$. Then, IVP (1) has a unique solution on $[0,1]$.

Proof: Consider $B_\gamma = \{u \in E : \|u\|_* < \gamma\}$ with $\gamma = \dfrac{\rho_2}{1-(1-\nu)\rho_1} > 0$, $\forall u \in B_\gamma$, we get

$$
\begin{aligned}
|Tu(t)| \le\ & |N_1|+|M_3|+\left(|N_2|+|M_4|\right)\left\{\int_0^1 (1-s)^{\alpha-2}\left|f\left(s,u(s)\right)-f(s,0)\right|ds+\int_0^1 (1-s)^{\alpha-2}\left|f(s,0)\right|ds\right\}\\
& +|N_5|\left\{\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,u(s)\right)-f(s,0)\right|ds+\int_0^1 (1-s)^{\alpha-1}\left|f(s,0)\right|ds\right\}\\
& +|M_4|\left\{\int_0^1 (1-s)^{\alpha-2}\left|f\left(s,u(s)\right)-f(s,0)\right|ds+\int_0^1 (1-s)^{\alpha-2}\left|f(s,0)\right|ds\right\}\\
& +|M_1|\left\{\int_0^t \left|f\left(s,u(s)\right)-f(s,0)\right|ds+\int_0^t \left|f(s,0)\right|ds\right\}\\
& +|M_2|\left\{\int_0^t (\mathrm{t}-s)^{\alpha-1}\left|f\left(s,u(s)\right)-f(s,0)\right|ds+\int_0^t (t-s)^{\alpha-1}\left|f(s,0)\right|ds\right\}\\
& +|N_6|\left\{\int_0^1 \left|g\left(s,u(s)\right)-g(s,0)\right|ds+\int_0^1 \left|g(s,0)\right|ds\right\}\\
& +\left(|N_3|+|M_5|\right)\left\{\int_0^1 \left|h\left(s,u(s)\right)-h(s,0)\right|ds+\int_0^1 \left|h(s,0)\right|ds\right\}\\
\le\ & |N_1|+|M_3|+\left(|N_2|+|N_4|+|M_4|\right)\|u\|\int_0^1 (1-s)^{\alpha-2}\left|a(s)\right|ds\\
& +\left(|N_2|+|N_4|+|M_4|\right)\int_0^1 (1-s)^{\alpha-2}\left|f(s,0)\right|ds\\
& +|N_5|\,\|u\|\int_0^1 (1-s)^{\alpha-1}\left|a(s)\right|ds+|N_5|\int_0^1 (1-s)^{\alpha-1}\left|f(s,0)\right|ds\\
& +|M_1|\,\|u\|\int_0^t \left|a(s)\right|ds+|M_1|\int_0^t \left|f(s,0)\right|ds\\
& +|M_2|\,\|u\|\int_0^t (t-s)^{\alpha-1}\left|a(s)\right|ds+|M_2|\int_0^t (t-s)^{\alpha-1}\left|f(s,0)\right|ds\\
& +|N_6|\,\|u\|\int_0^1 L_1 ds+|N_6|\int_0^1 \left|g(s,0)\right|ds\\
& +\left(|N_3|+|M_5|\right)\|u\|\int_0^1 L_2 ds+\left(|N_3|+|M_5|\right)\int_0^1 \left|h(s,0)\right|ds\\
\le\ & |N_1|+|M_3|+\left(|N_2|+|N_4|+|M_4|\right)\|u\|(1-v_1)+\left(|N_2|+|N_4|+|M_4|\right)R_1\\
& +|N_5|+|M_2|\|u\|(1-v_2)+\left(|N_5|+|M_2|\right)R_2+|M_2|\,\|u\|v_3+|M_1|R_3\\
& +|N_6|\|u\|v_4+|N_6|+|M_3|R+\left(|N_3|+|M_5|\right)|\,\|u\|v_5|+\left(|N_3|+|M_5|\right)+|M_4|R_3\\
\le\ & \Big(|M_1|+|M_2|+|M_4|+|N_2|+|N_4|+|N_5|+|N_6|L_1+\left(|N_3|+|M_5|\right)L_2\Big)(1-v)\,\|u\|\\
& +|N_1|+|M_3|+\left(|N_2|+|N_4|+|M_4|\right)R_1+|N_5|R_2+|M_1|K_1+|M_2|R_s\\
& +|N_6|K_2+\left(|N_3|+|M_5|\right)K_3
\end{aligned}
$$

$$= (1-\nu)\|u\|\rho_1 + \rho_2$$

$$\le \gamma,$$

which results $TB_\gamma \subseteq B_\gamma$. We have

$$\begin{aligned}
|Tu(t) - Tv(t)| \le{}& \left(|N_2| + |N_4|\right)\int_0^1 (1-s)^{\alpha-2}\left|f\left(s,u(s)\right) - f\left(s,v(s)\right)\right|ds \\
&+|N_5|\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,u(s)\right) - f\left(s,v(s)\right)\right|ds \\
&+|M_4|\, t\int_0^1 (1-s)^{\alpha-2}\left|f\left(s,u(s)\right) - f\left(s,v(s)\right)\right|ds \\
&+|M_1|\int_0^t \left|f\left(s,u(s)\right) - f\left(s,v(s)\right)\right|ds \\
&+|M_2|\int_0^t (t-s)^{\alpha-1}\left|f\left(s,u(s)\right) - f\left(s,v(s)\right)\right|ds \\
&+|N_6|\int_0^1 \left|g\left(s,u(s)\right) - g\left(s,v(s)\right)\right|ds \\
&+\left(|N_3| + |M_5|t\right)\int_0^1 \left|h\left(s,u(s)\right) - h\left(s,v(s)\right)\right|ds \\
\le{}& \left(|N_2| + |N_4|\right) t\int_0^1 (1-s)^{\alpha-2}|a(s)|\,|u(s) - v(s)|ds \\
&+|N_5|\int_0^1 (1-s)^{\alpha-1}|a(s)|\,|u(s) - v(s)|ds \\
&+|M_4|t\int_0^1 (1-s)^{\alpha-2}|a(s)|\,|u(s) - v(s)|ds \\
&+|M_1|\int_0^1 |a(s)|\,|u(s) - v(s)|ds \\
&+|M_2|\int_0^t (t-s)^{\alpha-1}|a(s)|\,|u(s) - v(s)|ds \\
&+|N_6|\int_0^1 L_1(1-\nu)|u(s) - v(s)|ds \\
&+\left(|N_3| + |M_5|t\right)\int_0^1 L_1(1-\nu)|u(s) - v(s)|ds \\
\le{}& \rho_1(1-\nu)\|u - v\|.
\end{aligned}$$

Since $\rho_1(1-\nu) < 1$, it yields T is contraction mapping.

□

Theorem 2: Assume that (H_1), (H_2) and $(H_4)-(H_6)$ hold, with

$$\rho_1 = (1-\nu)\left(|M_4| + |N_2| + |N_4| + |N_5| + |N_6|L_1 + \left(|N_3| + |M_5|\right)L_2\right) < 1. \quad (8)$$

Then IVP (1) has at least one solution on $[0,1]$.

Proof: Consider $B_\gamma = \{u \in E \,;\, \|u\| < \gamma\}$, where

$$\sqrt{\gamma} \geq \max\left\{4\eta_1\|p_1\|, 4\sqrt{\eta_2}, 4N_6\|P_2\|, 4\eta_3\|p_3\|\right\}$$

Now, we defines operator $\mathcal{A}$ and $\mathcal{B}$ on B_γ as

$$\mathcal{A}u(t) = M_1\int_0^t f\left(s,u(s)\right)ds + M_2\int_0^t (t-s)^{\alpha-1} f\left(s,u(s)\right)ds$$

$$\mathcal{B}u(t) = N_1 + M_3 t + \left(N_2+N_4\right)\int_0^1 (1-s)^{\alpha-2} f\left(s,u(s)\right)ds$$

$$+N_5\int_0^1 (1-s)^{\alpha-1} f\left(s,u(s)\right)ds + M_4 t\int_0^1 (1-s)^{\alpha-2} f\left(s,u(s)\right)ds$$

$$+N_6\int_0^1 g\left(s,u(s)\right)ds + \left(|N_3| + |M_5|t|\right)\int_0^1 h\left(s,u(s)\right)ds\ .$$

In first we show that $\mathcal{A}$ is a compact operator on B_γ.

Since f is continuous, the operator $\mathcal{A}$ is continuous

$$\|\mathcal{A}u\| = \sup_{t\in[0,1]}\left| M_1\int_0^t f\left(s,u(s)\right)ds + M_2\int_0^t (t-s)^{\alpha-1} f\left(s,u(s)\right)ds \right|$$

$$\leq \sup_{t\in[0,1]} M_1\int_0^t \left|f\left(s,u(s)\right)\right|ds + M_2 \sup_{t\in[0,1]}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,u(s)\right)\right|ds$$

$$\leq \sup_{t\in[0,1]} M_1\int_0^t P_1(s)\psi_1\left(|u(s)|\right)ds + M_2 \sup_{t\in[0,1]}\int_0^t (t-s)^{\alpha-1} P_1(s)\psi_1\left(|u(s)|\right)ds$$

$$\leq M_1\|P_1\|\psi_1\left(|u|\right) + M_2\frac{\|P_1\|\psi_1\left(|u|\right)}{\alpha}$$

$$\leq \left(M_1 + \frac{M_2}{\alpha}\right)\|P_1\|\psi_1\left(|u|\right)$$

$$\leq \left(M_1 + \frac{M_2}{\alpha}\right)\|P_1\|\gamma\,,$$

Which show that $\mathcal{A}$ is uniformly bounded on B_γ.

Form (H_1), we define $\varpi = \sup\left\{\left|f\,(t,u(t)\right|; t\in[0,1], u\in B_\gamma\right\}$. Now, for $t_1,t_2 \in (0,1), t_1 < t_2$, we have

$$\|\mathcal{A}u(t_1) - \mathcal{A}u(t_2)\| = \sup_{t\in[0,1]}\left| M_1\int_0^{t_2} f\left(s,u(s)\right)ds + M_2\int_0^{t_2}(t_2-s)^{\alpha-1} f\left(s,u(s)\right)ds \right.$$

$$\left. -M_1\int_0^{t_1} f\left(s,u(s)\right)ds - M_2\int_0^{t_1}(t_1-s)^{\alpha-1} f\left(s,u(s)\right)ds \right|$$

$$\leq |M_1| \sup_{t\in[0,1]} \int_{t_1}^{t_2} \left| f\left(s,u(s)\right) \right| ds + |M_2| \sup_{t\in[0,1]} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left| f\left(s,u(s)\right) \right| ds$$

$$+|M_2| \sup_{t\in[0,1]} \int_0^{t_1} \left((t_s - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right) \left| f\left(s,u(s)\right) \right| ds$$

$$\leq \varpi |M_1| (t_2 - t_1) + \varpi \frac{|M_2|}{\alpha} (t_2 - t_1)^{\alpha} + \varpi \frac{|M_2|}{\alpha} \left| t_2^{\alpha} - t_1^{\alpha} - (t_2 - t_1)^{\alpha} \right|$$

that tends to zero as $t_2 \to t_1$. Therefore, $\mathcal{A}$ is relatively compact on B_γ. By the Arzela-Ascoli theorem $\mathcal{A}$ is compact on B_γ. Now, for $u,v \in B_\gamma$, we show that $\mathcal{A}u + \mathcal{B}u \in B_\gamma$.

$$\|\mathcal{A}u + \mathcal{B}u\| = \sup_{t\in[0,1]} \Big| M_2 \int_0^t (t-s)^{\alpha-1} f\left(s,u(s)\right) ds + M_1 \int_0^t f\left(s,u(s)\right) ds$$

$$+N_1 + M_3 t + (N_2 + N_4) \int_0^1 (1-s)^{\alpha-2} f\left(s,u(s)\right) ds$$

$$+N_5 \int_0^1 (1-s)^{\alpha-1} f\left(s,u(s)\right) ds + M_4 t \int_0^1 (1-s)^{\alpha-2} f\left(s,u(s)\right) ds$$

$$+N_6 \int_0^1 g\left(s,u(s)\right) ds + \left(N_3 + M_5 t\right) \int_0^1 h\left(s,u(s)\right) ds \Big|$$

$$+|M_2| \sup_{t\in[0,1]} \int_0^t (t-s)^{\alpha-1} \left| f\left(s,u(s)\right) \right| ds + |M_1| \sup_{t\in[0,1]} \int_0^t \left| f\left(s,u(s)\right) \right| ds$$

$$+ \sup_{t\in[0,1]} \left(|N_1| + |M_3| t \right) + \left(|N_2| + |N_4| \right) \int_0^1 (1-s)^{\alpha-2} \left| f\left(s,u(s)\right) \right| ds$$

$$+|N_5| \int_0^1 (1-s)^{\alpha-1} \left| f\left(s,u(s)\right) \right| ds + \sup_{t\in[0,1]} |M_4| t \int_0^1 (1-s)^{\alpha-2} \left| f\left(s,u(s)\right) \right| ds$$

$$+N_6 \int_0^1 \left| g\left(s,u(s)\right) \right| ds + \sup_{t\in[0,1]} \left(N_3 + M_5 t\right) \int_0^1 \left| h\left(s,u(s)\right) \right| ds$$

$$\leq \left(\frac{|M_2|}{\alpha} + |M_1| \right) \| p_1 \| \psi_1\left(|u|\right) + |N_1| + |M_3| + \frac{|N_2| + |N_4|}{\alpha - 1} \| p_1 \| \psi_1\left(|u|\right)$$

$$+ \frac{|N_5|}{\alpha} \| p_1 \| \psi_1\left(|u|\right) + \frac{|M_4|}{\alpha - 1} \| p_1 \| \psi_1\left(|u|\right) + |N_6| \| p_2 \| \psi_2\left(|u|\right)$$

$$+ \left(|N_3| + |M_5| \right) \| p_3 \| \psi_3\left(|u|\right)$$

$$= \left(|M_2| \alpha + |M_1| + \frac{|N_1| + |N_4|}{\alpha - 1} + \frac{|N_5|}{\alpha} + \frac{|M_4|}{\alpha - 1} \right) \| p_1 \| \psi_1\left(|u|\right)$$

$$+ |N_1| + |M_3| + |N_6| \| p_2 \| \psi_2\left(|u|\right) + \left(|N_3| + |M_5| \right) \| p_3 \| \psi_3\left(|u|\right)$$

$$\leq \eta_1 \| p_1 \| \sqrt{|u|} + \eta_2 |N_6| \| p_2 \| \sqrt{|u|} + \eta_3 \| p_3 \| \sqrt{|u|}$$

$$\le \frac{\gamma}{4}+\frac{\gamma}{4}+\frac{\gamma}{4}+\frac{\gamma}{4}=\gamma.$$

Furthermore, from (3), similar to Theorem 3 we conclude that β is contraction mapping. Thus, all assumptions of Krasnoselskii's fixed point theorem are satisfied. Therefore, BVP (1) has at least one solution on $[0,1]$. □

4. Conclusions

In this paper, we have successfully applied the contraction mapping principle and Krasnoselskii's fixed point theorem to obtain some new existence and uniqueness results for fractional integral boundary value problems of Atangana-Baleanu type.

Finally, it should be added that the suggested approach has the potentials to be applied to obtain results for other similar nonlinear problems of fractional order.

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